THEORETICAL FRAMEWORKS FOR TESTING RELATIVISTIC GRAVITY.

V. POST-NEWTONIAN LIMIT OF ROSEN'S THEORY*

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ABSTRACT

The post-Newtonian limit of Rosen's theory of gravity is evaluated and is shown to be identical to that of general relativity, except for the PPN parameter α_2 (which is related to the difference in propagation speeds for gravitational and electromagnetic waves). Both the value of α_2 and the value of the Newtonian gravitational constant depend on the present cosmological structure of the Universe. If the cosmological structure has a specific (but presumably special) form, the Newtonian gravitational constant assumes its current value, α_2 is zero, the post-Newtonian limit of Rosen's theory is identical to that of general relativity--and standard solar system experiments cannot distinguish between the two theories.

I. INTRODUCTION

Nathan Rosen (1973, 1974) has recently devised a new theory of gravity. It is a bimetric theory in the sense that it possesses two metric fields--a flat metric η_{ij} with "light cones" along which weak gravitational waves propagate, and a curved metric g_{ij} with "light cones" along which light propagates. It is also a "metric theory" in the sense that the Einstein equivalence principle holds in the local Lorentz frames of the "physical metric" g_{ij} (cf. § 39.2 of Misner, Thorne, and Wheeler 1973--cited henceforth as MTW). The field equations of the theory are derivable from a variational principle. In forthcoming papers Rosen and his colleagues will use the theory to analyze cosmology and neutron stars.

In this paper we evaluate the post-Newtonian limit of Rosen's theory by considering an isolated system with weak internal gravity (such as the solar system). We express our results in the language of the Nordtvedt-Will Parametrized Post-Newtonian (PPN) Formalism (see Chapter 39 of MTW; also the review by Will 1974). At any particular epoch in the evolution of the Universe, the boundary conditions in the asymptotically flat region far outside the system are determined by the cosmological structure of the Universe at that epoch. We assume boundary conditions which are appropriate for a homogeneous, isotropic cosmological model. We find that the Newtonian gravitational constant and the value of the PPN parameter α_2 depend on the cosmological boundary values. For a particular choice of cosmological boundary values, the Newtonian gravitational constant assumes its present value,

α_2 is zero, and the post-Newtonian limit is identical to that of general relativity.

The notation and format of this paper will be the same as in Ni's (1972) compendium of metric theories of gravity.

II. EVALUATION OF POST-NEWTONIAN LIMIT

a. Principal References: Rosen (1973, 1974)

b. Gravitational Fields: $\mathbf{g}$, $\boldsymbol{\eta}$

A semicolon ";" and a slash "|" denote covariant derivatives with respect to $\mathbf{g}$ and $\boldsymbol{\eta}$, respectively.

c. Field Equations:

$$\tfrac{1}{2}\eta^{kl}\, g_{ij|kl} - \tfrac{1}{2}\eta^{kl}\, g^{mn} g_{mi|k} g_{nj|l} = -8\pi \frac{\sqrt{-g}}{\sqrt{-\eta}}\, G_o \left(T_{ij} - \tfrac{1}{2} g_{ij}\, g^{kl} T_{kl}\right) \qquad (1)$$

$$\mathbf{R}\mathrm{iem}(\boldsymbol{\eta}) = 0 \qquad (2)$$

The field equations are written in units with the speed of light $c = 1$. G_o is a coupling constant with the dimensions of the Newtonian gravitational constant. The field equations are invariant under the rescaling given by $\eta_{ij} \to \eta^*_{ij} = A\,\eta_{ij}$, $g_{ij} \to g^*_{ij} = g_{ij}$, and $G_o \to G^*_o = AG_o$, where A is a constant; therefore, G_o can be chosen to have any convenient value. We choose G_o to be the Newtonian gravitational constant in the solar system today. In his original presentation Rosen chose units such that $G_o = 1$.

d. The Post-Newtonian Solution:

Consider an isolated system with weak internal gravity. As in general relativity (MTW, Chapter 19), so also in Rosen's theory, there

exist coordinate systems $\{x^i\}$ in which g_{ij} is asymptotically Minkowskiian:

$$\|g_{ij}\| \to \mathrm{diag}(1, -1, -1, -1) \qquad \text{far from source.} \tag{3}$$

However, the cosmological structure of the Universe at the epoch of interest will typically force η_{ij} to have a <u>non</u>-Minkowskiian form in these coordinates. To avoid this complication, we shall compute in coordinates $\{x^{i'}\}$ with

$$\|\eta_{i'j'}\| = \mathrm{diag}(1, -1, -1, -1) \ , \tag{4a}$$

$$\|g_{i'j'}\| \to \|\overset{(B)}{g}_{i'j'}\| \neq \mathrm{diag}(1, -1, -1, -1) \quad \text{far from source.} \tag{4b}$$

By definition the cosmological background metric (eq. [4b]) changes only on cosmological time scales. Since cosmological time scales are so much longer than the dynamical time scales of a system such as the solar system, we can ignore the variation of the cosmological background metric in calculating the post-Newtonian limit.

We can simplify the cosmological background metric, without destroying the form (eq. [4a]) of $\eta_{i'j'}$, by performing an appropriate Lorentz transformation. More specifically, by an appropriate boost we can set $\overset{(B)}{g}_{o'\alpha'} = 0$, and by a subsequent rotation we can diagonalize $\overset{(B)}{g}_{\alpha'\beta'}$, thereby obtaining

$$\|\overset{(B)}{g}_{i'j'}\| = \mathrm{diag}(c_o, -c_1, -c_2, -c_3). \tag{5}$$

In general the four constants c_j will be different. However, for the simple case of a homogeneous, isotropic cosmological model, a further simplification of the cosmological background metric is possible (Rosen 1975, Caves 1975). In the universal rest frame of such a cosmological

model (frame comoving with the cosmological fluid; frame in which the black-body background radiation is isotropic), equation (5) holds with $c_1 = c_2 = c_3$. Henceforth, we will restrict ourselves to this case:

$$\|g^{(B)}_{i'j'}\| = \mathrm{diag}(c_o, -c_1, -c_1, -c_1) \quad . \tag{6}$$

The simple form of equation (6) will be maintained under boosts if and only if $c_o = c_1$. Therefore, if $c_o \neq c_1$, assuming equation (6) is equivalent to assuming that our calculation of the post-Newtonian limit is carried out in the universal rest frame.

We first solve the field equations (1) to Newtonian order. The solution is

$$g_{o'o'} = c_o[1 - 2c_1(c_oc_1)^{1/2}G_oU'] \ , \tag{7a}$$

$$g_{o'\alpha'} = 0 \ , \tag{7b}$$

$$g_{\alpha'\beta'} = -c_1\delta_{\alpha\beta} \ , \tag{7c}$$

where U' is the Newtonian potential:

$$U'(\underset{\sim}{x}',t') = \int \frac{\rho(\underset{\sim}{\xi}',t')}{|\underset{\sim}{x}'-\underset{\sim}{\xi}'|}\, d\underset{\sim}{\xi}' \quad , \tag{8}$$

where ρ is the rest mass density in the <u>proper</u> rest frame of the material [frame with $\|g_{\hat{i}\hat{j}}\| = \mathrm{diag}(1, -1, -1, -1)$ and $u^{\hat{i}} = \delta^i_o$, where u^i is the 4-velocity of the material].

In order to compare with the correct Newtonian limit, we introduce coordinates $\{x^i\}$ in which g_{ij} is asymptotically Minkowskiian:

$$x^{0} = c_{0}^{\frac{1}{2}} \; x^{0'} \quad , \tag{9a}$$

$$x^{\alpha} = c_{1}^{\frac{1}{2}} \; x^{\alpha'} \quad . \tag{9b}$$

In these new "unprimed" coordinates the metric (eqs. [7]) becomes

$$g_{00} = 1 - 2(c_0 c_1)^{\frac{1}{2}} \, G_0 U \quad , \tag{10a}$$

$$g_{0\alpha} = 0 \quad , \tag{10b}$$

$$g_{\alpha\beta} = -\delta_{\alpha\beta} \quad , \tag{10c}$$

where we have rescaled the New   potential appropriately, i.e.,

$$U(\underset{\sim}{x},t) = c_1 U'(\underset{\sim}{x}',t') = \int \frac{\rho(\underset{\sim}{\xi},t)}{|\underset{\sim}{x} - \underset{\sim}{\xi}|} \, d\underset{\sim}{\xi} \quad . \tag{11}$$

It follows from equations (10) that, at any given epoch in the evolution of the Universe, the Newtonian gravitational constant is given by

$$G = (c_0 c_1)^{\frac{1}{2}} \, G_0 \quad . \tag{12}$$

For the remainder of the calculation we choose units such that, at the particular epoch of interest, $G = 1$.

We now use standard procedures to solve the field equations (1) to post-Newtonian order. In the standard post-Newtonian gauge the solution is

$$g_{0'0'} = c_0[\, 1 - 2c_1 U' + 2c_1^2 (U')^2 - 4\Phi' \,] \; , \tag{13a}$$

$$g_{o'\alpha'} = c_1^2 \left[\left(4 - \frac{1}{2}\frac{c_o}{c_1}\right) V'_\alpha + \frac{1}{2}\frac{c_o}{c_1} W'_\alpha \right] \quad , \tag{13b}$$

$$g_{\alpha'\beta'} = -c_1 \, \delta_{\alpha\beta}[1 + 2c_1 U'] \quad , \tag{13c}$$

where the potentials V'_α , W'_α are as on page 1085 of MTW, where

$$\Phi'(\underset{\sim}{x}', t') = \int \frac{\rho(\underset{\sim}{\xi}', t')\phi'(\underset{\sim}{\xi}', t')}{|\underset{\sim}{x}' - \underset{\sim}{\xi}'|} \, d\underset{\sim}{\xi}' \quad , \tag{14a}$$

$$\phi' = \frac{c_1^2}{c_o}(v')^2 + c_1^2 U' + \frac{1}{2}c_1 \Pi + \frac{3}{2} c_1 \frac{p}{\rho} \quad , \tag{14b}$$

and where p and Π are pressure and specific internal energy in the proper rest frame of the material and $\underset{\sim}{v}'$ is the coordinate velocity of the material, $\underset{\sim}{v}' = d\underset{\sim}{x}'/dx^{o'}$.

In order to compare with the PPN formalism, we now transform the metric (eqs. [13]) to the "unprimed" coordinates (eqs. [9]). After rescaling the various potentials accordingly, e.g.,

$$V_\alpha(\underset{\sim}{x}, t) = c_1(c_1/c_o)^{\frac{1}{2}} V'_\alpha(\underset{\sim}{x}', t') \quad , \tag{15}$$

the transformed solution is

$$g_{oo} = 1 - 2U + 2U^2 - 4\Phi \quad , \tag{16a}$$

$$g_{o\alpha} = \left(4 - \frac{1}{2}\frac{c_o}{c_1}\right) V_\alpha + \frac{1}{2}\frac{c_o}{c_1} W_\alpha \quad , \tag{16b}$$

$$g_{\alpha\beta} = -\delta_{\alpha\beta}(1 + 2U) \quad , \tag{16c}$$

where the potentials U, V_α, W_α, Φ are all as on page 1085 of MTW. By

comparing equations (16) with the standard PPN metric (see, e.g., MTW §39.8 and eq. [4] of Will 1973), we obtain the following values for the PPN parameters of Rosen's theory:

$$\gamma = \beta = 1; \quad \zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \zeta_W = \alpha_1 = \alpha_3 = 0 ;$$

$$\alpha_2 = (c_o/c_1) - 1 \quad . \tag{17}$$

The PPN parameters of Rosen's theory are identical to those of general relativity, except for the preferred frame parameter α_2 which is nonzero whenever $c_o \neq c_1$. The value of α_2 can be related to the relative propagation speeds of electromagnetic and weak gravitational waves in Rosen's theory. In the "unprimed" coordinates the physical metric has the asymptotic form $\|g_{ij}\| \to \mathrm{diag}(1, -1, -1, -1)$, while the flat metric $\boldsymbol{\eta}$ has the form $\|\eta_{ij}\| = \mathrm{diag}(c_o^{-1}, -c_1^{-1}, -c_1^{-1}, -c_1^{-1})$. Thus the vacuum, linearized equations for g_{ij} (wave equations for weak gravitational waves) take the form

$$(c_o/c_1)\, g_{ij,oo} - \nabla^2 g_{ij} = 0 \quad , \tag{18}$$

whose solution is a wave propagating with speed $v_g = (c_1/c_o)^{1/2}$. In the "unprimed" coordinates, electromagnetic waves propagate with speed unity. Thus the PPN parameter α_2 measures the relative difference in speeds (as measured by an observer at rest in the universal rest frame) between electromagnetic and gravitational waves in Rosen's theory, and is given by

$$\alpha_2 = (1/v_g^2) - 1 \tag{19}$$

(see Will 1971 for another example of a theory with this property). Earth-tide measurements place an upper limit on α_2 given by (Will 1971)

$$|\alpha_2| = |(1/v_g^2) - 1| < 3 \times 10^{-2} . \tag{20}$$

Equations (12) and (17) show that, in Rosen's theory, the values of both the Newtonian gravitational constant and the PPN parameter α_2 depend on the cosmological boundary values. In general, both will change as the Universe evolves. In a separate paper, Caves (1975) will analyze the homogeneous, isotropic cosmologies of Rosen's theory. He will show that there exists a special epoch in such cosmologies at which $c_o = c_1 = 1$. At that epoch the Newtonian gravitational constant has its current value, and the PPN parameters are precisely the same as those of general relativity.

Notice that the constants c_o and c_1 can be written in terms of scalars constructed from $\boldsymbol{\eta}$ and $\mathbf{g}$:

$$c_o + 3c_1 = \eta^{\alpha\beta} \; g_{\alpha\beta} \quad , \tag{21a}$$

$$\frac{1}{c_o} + \frac{3}{c_1} = \eta_{\alpha\beta} \; g^{\alpha\beta} \quad , \tag{21b}$$

where the expressions on the right are to be evaluated in the asymptotically flat region outside the solar system.

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